The dynamics of plant disease models with continuous and impulsive cultural control strategies

Xinzhu Meng, Zhenqing Li

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ABSTRACT

Plant disease mathematical models including continuous cultural control strategy and impulsive cultural control strategy are proposed and investigated. This novel theoretical framework could result in an objective criterion on how to control plant disease transmission by replanting of healthy plants and removal of infected plants. Firstly, continuous replanting of healthy plants and removing of infected plants is taken. The existence and stability of disease-free equilibrium and positive equilibrium are studied and continuous cultural control strategy is given. Secondly, plant disease model with impulsive replanting of healthy plants and removing of infected plants is also considered. Using Floquet’s theorem and small amplitude perturbation, the sufficient conditions under which the infected plant free periodic solution is locally stable are obtained. Moreover, permanence of the system is investigated. Under certain parameter spaces, it is shown that a nontrivial periodic solution emerges via a supercritical bifurcation. Finally, our findings are confirmed by means of numerical simulations. The modeling methods and analytical analysis presented can serve as an integrating measure to identify and design appropriate plant disease control strategies.

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1. Introduction

Plant viruses or pathogens are an important constraint to crop production worldwide, and cause serious losses in yield and quality of cultivated plants. Therefore, plant disease are widely regarded. The 10th International Plant Virus Epidemiology Symposium on the theme ‘Controlling Epidemics of Emerging and Established Plant Virus Diseases—the Way Forward’ was held in India. In this symposium, many novel and interesting viewpoints and ways on plant virus epidemiology were offered and elaborated. Several plant diseases caused by plant viruses in cassava (Manihot esculenta) and sweet potato (Ipomoea batatas) are among the main constraints to sustainable production of these vegetatively propagated staple food crops in lesser-developed countries (Thresh and Cooter, 2005; Dubern, 1994; Gibson et al., 2004). Epidemics of many polycyclic plant diseases caused by ascomycete fungal pathogens are initiated by wind-borne ascospores transported into the crop in autumn (Inman et al., 1999) and subsequently develop further through cycles of splash dispersed conidia (Madden et al., 1996; Evenhuis et al., 1997). Sharka is one of the most severe diseases damaging stone-fruit trees including apricot, cherry, peach and plum. The causative agent is plum pox potyvirus (PPV) that is naturally transmitted by aphids. No effective measures have been found to restrict PPV outbreaks and it has spread across European borders to the Middle East, North Africa, and North and South America. It seems that the best measure is to remove the diseased stone-fruit trees. Therefore, farmers have been evolving practices for combating the various plagues suffered by crops and trees, and growing understanding of the interactions between pathogen and host has enabled us to develop a wide array of measures for the control of specific plant diseases.

Such experiences have led to the development of integrated management concepts for virus diseases that combine available host resistance, cultural, chemical and biological control measures. Examples of how epidemiological information can be used to develop effective integrated disease management (IDM) strategies for diverse situations have been described (Jones, 2001, 2004). IDM involves the selection and application of a wide range of control strategies that minimize losses and maximize returns. A cultural control strategy including replanting, and/or removing diseased plants is a widely accepted treatment for plant epidemics which involves periodic inspections followed by removal of the detected infected plants (Fishman and Marcus, 1984; Jeger et al., 2004).
Mathematical models of plant–virus (or pathogens) disease epidemics were developed to provide a detailed exposition on how to describe, analyze, and predict epidemics of plant disease for the ultimate purposes of developing and testing control strategies and tactics for crop protection (Grilli and Holt, 2000; Chan and Jeger, 1994; Holt and Chancellor, 1997). A simple model for plant disease with a continuous cultural control strategy, such as replanting and removing, is as follows:

\[
\begin{align*}
\frac{dS(t)}{dt} &= \rho - \beta S(t)I(t) + \mu S(t) + \alpha l(t), \\
\frac{dl(t)}{dt} &= \beta S(t)I(t) - \frac{(d+r+\omega)o}{C_0}l(t), \\
S(t^+) &= S(t) + \rho, \\
l(t^+) &= (1-r)l(t),
\end{align*}
\]

(1.1)

where \(S(t), l(t)\) denote the number of susceptible and infected plants, respectively. \(\beta\) is the transmission rate, \(\alpha\) denotes potentially density dependent, \(\rho\) either denotes harvest time rate or the end of reproductive lifetime of plants, \(\rho\) represents the total rate at which the susceptible plants enter the system, \(r\) is the removal rate for the infected plants, \(\omega\) is the recovery rate of the cured diseased plants, and the infected plants suffer an extra disease-related death with constant rate \(d\). The detailed biological background and model development can be found in Chan and Jeger (1994). Further, a model for the temporal spread of an epidemic in a closed plant population with periodic removals of infested plants has been considered by Fishman et al. (1983) with an application to the spread of citrus tristeza virus disease. Their model helped in evaluating policies of controlling the disease and could be also modified to simulate other plant epidemics with periodic treatments. Fishman et al. (1983) proposed two types of model with Logistic growth and periodic removing of infected trees, aimed at eradicating them. Therefore, one of the main purposes of the models described in this paper is to extend the temporal spread of an epidemic in a closed plant population with periodic removals of infected and removed for the infected plants, and investigate how they affects the dynamical behaviors of unforced continuous system.

The organization of this paper is as follows. In Section 2, we investigate dynamical behaviors of system (1.1) with continuous replanting and removing and give plant disease continuous control strategies. In this section, stability of two steady states of system (1.1) is investigated. In Section 3, we investigate dynamical behaviors of system (1.2) with impulsive replanting and removing and give plant disease control strategies. In this section, stability of disease-free periodic solution and permanence of system (1.2) are investigated, respectively. In Section 4, we investigate existence of positive periodic solution and bifurcation of system (1.2). Finally, we give numerical analysis and biological conclusions to show our main results.

2. Plant disease continuous control for system (1.1)

In this section, we consider system (1.1) with continuous replanting and removing and without impulsive effect. Obviously, we have \(S(t) \geq 0, I(t) \geq 0\) for \(S(0) \geq 0, I(0) \geq 0, t \geq 0\). Define \(N(S, I) = S(t) + I(t)\),

\[
\frac{dN(t)}{dt} = \rho - \beta S(t)I(t) - (r + d)I(t) \\
\leq \rho - \gamma_1(S(t) + I(t)),
\]

where \(\gamma_1 = \min(\mu, (r + d))\). Hence, system (1.1) is uniformly bounded.

An equilibrium point of system (1.1) satisfies the system

\[
\begin{align*}
\rho - \beta S(t)I(t) - \frac{(d+r+\omega)o}{C_0}l(t) &= 0, \\
\beta S(t)I(t) - \frac{(d+r+\omega)l(t)}{C_0} &= 0.
\end{align*}
\]

(2.1)

It can be seen that system (2.1) has a disease-free equilibrium of the form \(E_0 = (\rho/\mu, 0)\). We start by analyzing the behavior of the system of (1.1) near \(E_0\). The characteristic equation of the linearization of (1.1) \(E_0\) is

\[
\begin{pmatrix}
-\mu - \lambda & \beta \rho \\ 0 & \mu + \alpha \rho - (d + r + \omega) - \lambda
\end{pmatrix}.
\]

It is easy to see that \(\lambda_1 = -\mu < 0\) and \(\lambda_2 = \beta \rho/(\mu + \alpha \rho) - (d + r + \omega) < 0\) is equivalent to

\[
\frac{\beta \rho}{\mu + \alpha \rho} < (d + r + \omega).
\]

(2.2)

So the boundary steady state \(E_0\) is locally stable if (2.2) holds.

Define the threshold value

\[
R_0 = \frac{\beta \rho}{(\mu + \alpha \rho)(d + r + \omega)}.
\]

The above results show that the disease-free equilibrium \(E_0\) is locally asymptotically stable if \(R_0 < 1\), and \(E_0\) is unstable if \(R_0 > 1\). If \(R_0 < 1\), then system (1.1) has a unique steady state \(E_0\). It follows from the uniform boundedness and Fig. 1 that the disease-free
The plants density initial value is \((S_0, I_0)\) with \(r=0.8\); (b) time series of \(S(t)\) with \(r=0.8\); (c) phase portrait of \(S(t)\) and \(I(t)\) with \(r=0.4\). Suppose the unit of the plants density \(S\) and \(I\) is thousand and the unit of time is day.

![Illustration of basic behavior of solutions of the model (1.1)](image)

The Jacobian matrix at \(E^*(S^*, I^*)\) is

\[
\begin{pmatrix}
-\mu \beta S^* \\
\frac{\beta S^* I^*}{1 + 2S^*} \\
\frac{\beta S^*}{1 + 2S^*} \\
\end{pmatrix} 
\begin{pmatrix}
\omega - \beta S^* \\
\frac{\beta S^*}{1 + 2S^*} \\
\frac{\beta S^*}{1 + 2S^*} (d + r + \omega) \\
\end{pmatrix}
\]

After a few computations, we know that the real parts of eigenvalues are negative if and only if \(R_0 > 1\). Hence, as long as \(R_0 > 1\) holds, the positive equilibrium of system (1.1) is locally asymptotically stable.

Further, let \(P(S_I), Q(S_I)\) denote the right side of (1.1), respectively, then we have

\[
P(S_I) = \rho - \frac{\beta S(t)(t)}{1 + 2S(t)} - \mu S(t) + \omega I(t), \quad Q(S_I) = \frac{\beta S(t)(t)}{1 + 2S(t)} - (d + r + \omega)I(t).
\]

Set Dulac function

\[
R(S_I) = l^{-1}
\]

Then we have

\[
\frac{\partial BQ}{\partial S} - \frac{\partial BP}{\partial I} = -\frac{\beta}{(1 + 2S)^2} - \mu l^{-1} < 0.
\]

Hence, there is not limit-cycle in \(R^2\), so, the positive equilibrium \(E^*(S^*, I^*)\) is globally asymptotically stable (see Fig. 1c).

**Theorem 2.0.** (i) The system (1.1) has a unique equilibrium \(E_0\) which is globally asymptotically stable if \(R_0 < 1\). (ii) If \(R_0 > 1\), system (1.1) has a unique positive equilibrium \(E^*(S^*, I^*)\) which is globally asymptotically stable, and \(E_0\) is unstable.

We have developed and extended the epidemic model with continuous cultural control to include the impulsive cultural control strategy such as replanting and/or removing diseased plants. Therefore, we give our main results as follows.

### 3. Plant disease impulsive control for system (1.2)

By the basic theories of impulsive differential equations (Bainov and Simeonov, 1993; Lakshmikantham et al., 1989), the solution of system (3.1) is unique and piecewise continuous in \((\tau, \tau + 1), n \in N\) for any initial values in \(R^2\).

#### 3.1. Disease-free periodic solution

**Lemma 3.0.** There exists a constant \(L > 0\) such that \(S(t), I(t) \leq L\) for each solution of (1.2) with \(t\) large enough.

**Proof.** Let \(V(t) = S(t) + I(t)\). Since \(0 < r < 1, (\nu t + \tau) < l\), thus \(V(\nu t + \tau) \leq V(\nu t) + \rho\) for \(V(t)\), the following inequalities hold for \(\tau \geq 0\),

\[
\begin{align*}
\dot{V}(t) & \leq -\gamma_2(S(t) + I(t)) = -\gamma_2 V(t), \quad t \neq \pi t, \\
\dot{V}(\pi t) & \leq V(\pi t) + \rho, \quad t = \pi t,
\end{align*}
\]

where \(\gamma_2 = \min(\mu, \omega, d)\). Hence, we have

\[
\begin{align*}
V(t) \leq V(0)e^{-\gamma_2 t} + \frac{\rho}{1 - e^{-\gamma_2 t}} & \rightarrow \frac{\rho}{1 - e^{-\gamma_2 t}} \text{ for } t \rightarrow \infty.
\end{align*}
\]

So \(V(t)\) is uniformly ultimately bounded. Hence, by the definition of \(V(t)\), for any \(c > 0\) there exists a constant \(L = \rho / (1 - e^{-\gamma_2 t}) + c\) such that \(S(t), I(t) \leq L\) for each solution of (1.2) with \(t\) large enough. The proof is completed. □

In the following we shall prove the disease-free periodic solution is stable if it exists. For this purpose, we give firstly
some basic properties of the following subsystem:

\[
\begin{aligned}
\frac{dS(t)}{dt} &= -\mu S(t), \quad t \neq n\tau, \\
\frac{dI(t)}{dt} &= \mu S(t) - \mu I(t) - B \beta S(t) I(t)/(1 + 2 S(t)) + \gamma I(t) - \beta S(t) I(t)/(1 + 2 S(t)), \\
S(t^n) &= S(t^{n+1}) + \rho, \quad t = n\tau.
\end{aligned}
\] (3.1)

We can find a unique positive periodic solution \( \bar{S}(\tau) = \rho \exp(-\mu t)/(1 + 2 \bar{S}(\tau)), t \in (n\tau, (n+1)\tau) \). Similar to Liu and Chen (2003), it can be shown that \( \bar{S}(\tau) \) is globally asymptotically stable by using stroboscopic map. As a consequence, system (3.1) always has a disease-free periodic solution \( (\bar{S}(\tau), 0) \). The local stability of the periodic solution may be determined by considering the behavior of small amplitude perturbations of the solution.

The Jacobi matrix \( A(t) \) at \( (\bar{S}(\tau), 0) \) and matrix \( B \) are as follows:

\[
A(t) = \begin{bmatrix}
-\mu & -B \beta S(t) \\
0 & -\mu + 2 \beta S(t)
\end{bmatrix},
B = \begin{bmatrix}
1 & 0 \\
0 & 1 - r
\end{bmatrix}.
\]

We can calculate the monodromy matrix \( M(\tau) \)

\[
M(\tau) = Be^{\int_{0}^{\tau} A(t) dt} = \begin{bmatrix}
e^{-\mu \tau} & \frac{\mu S(t)}{1 + 2 \bar{S}(t)} (1 - r) \exp\left( \int_{0}^{\tau} \frac{\beta S(t)}{1 + 2 \bar{S}(t)} dt - (d + \omega) \tau \right) 
\end{bmatrix},
\]

where \( \int_{0}^{\tau} \frac{\beta S(t)}{1 + 2 \bar{S}(t)} dt = \beta /2 \mu \ln(1 - e^{-\mu \tau} + 2 \rho / \beta / \mu), \) and there is no need to calculate the exact form of \( * \) since it is not required for the following analyses. The Floquet multipliers are \( \lambda_1 = \exp(-\mu \tau) \) and

\[
\lambda_2 = (1 - r) \exp\left( \int_{0}^{\tau} \frac{\beta S(t)}{1 + 2 \bar{S}(t)} dt - (d + \omega) \tau \right),
\]

respectively. Obviously, \( \lambda_1 < 1 \), so the stability of \( (\bar{S}(\tau), 0) \) is decided by whether

\[
\lambda_2 = (1 - r) \exp\left( \int_{0}^{\tau} \frac{\beta S(t)}{1 + 2 \bar{S}(t)} dt - (d + \omega) \tau \right) < 1,
\]

that is, when

\[
\frac{\beta}{2 \mu (d + \omega) \tau - \ln(1 - r)} \ln\left( \frac{1 - e^{-\mu \tau} + 2 \rho}{1 - e^{-\mu \tau} + 2 \rho e^{-\mu \tau}} \right) < 1,
\]

the disease-free solution is locally stable.

Define the threshold value

\[
R_1 = \frac{\beta}{2 \mu (d + \omega) \tau - \ln(1 - r)} \ln\left( \frac{1 - e^{-\mu \tau} + 2 \rho}{1 - e^{-\mu \tau} + 2 \rho e^{-\mu \tau}} \right).
\]

**Theorem 3.0.** If \( R_1 < 1 \), then the disease-free periodic solution \( (\bar{S}(\tau), 0) \) of system (1.2) is locally stable.

We can show the disease-free periodic solution \( (\bar{S}(\tau), 0) \) of system (1.2) is globally asymptotically stable by numerical simulations (see Fig. 2), but it is very difficult to prove theoretically the global asymptotical stability now. Hence, we will do it in the future.

### 3.2. Impulsive control strategy with Economic Threshold

The result of Theorem 3.0 implies that the infected plants can be completely eradicated if the impulsive period of control strategy application is sufficiently small or the removal rate for the infected plants is sufficiently large, i.e. the infected plant free periodic solution is stable. However, a good plant disease control program should reduce the infected plants to levels acceptable to the public. That is, we do not need to eradicate the infected plants but to remove the infected plants under an Economic Threshold (ET). Therefore, we consider following how to remove the infected plants under an Economic Threshold (ET). In practice, on the one hand it is usually required in agriculture and forestry that we
control the pest density to be lower than the economic injury level in a finite given time before harvest. On the other hand the control is considered in an infinite time.

We first consider the ET strategy for a finite time interval now.

Using Lemma 3.0, it follows from the second and fourth equations of (1.2) that

\[
\frac{dl}{dt} \leq \left( \frac{\beta L}{1+2\lambda} \right) l(t), \quad t \neq n\tau, \\
\frac{l(t^+)}{l(t^-)} = (1-r)l(t), \quad t = n\tau.
\]  

(3.2)

Consider the following impulsive differential equation

\[
\frac{du}{dt} = \left( \frac{\beta L}{1+2\lambda} \right) u(t), \quad t \neq n\tau, \\
u(t^+)= (1-r)u(t), \quad t = n\tau, \\
u(0)=l(0) > 0.
\]  

(3.3)

Suppose that the times of impulsive removing the infected plants taken (m is a positive integer) in a given time period τ satisfies 0 < t < 2τ ... < mτ ≤ T0, then the model (3.3) becomes

\[
\frac{du}{dt} + \sigma u(t), \quad t \neq n\tau, \\
u(t^+) = (1-r)u(t), \quad t = n\tau, \\
u(0) = h, \quad u(T_0) \leq ET.
\]  

(3.4)

where σ = βL/(1+2λ) – (d+ω) > 0, 0 ≤ t ≤ T0, h and ET are positive constants. Without loss of generality, by real-life biological meanings we assume that u0(T0) > ET when r = 0.

We can calculate the solution of the first two equations of system (3.4)

\[
u(t) = u(0)((1-r)e^{\sigma(T_0-t)})e^{\sigma(t-n\tau)}, \quad t \in (n\tau, (n+1)\tau].
\]  

Then, we have

\[
u(T_0) = u(0)((1-r)e^{\sigma(T_0-T_0)})e^{\sigma(T_0-n\tau)}, \quad T_0 \in (n\tau, (n+1)\tau].
\]  

(3.5)

In order to remove the infected plants under an Economic Threshold (ET), suppose (1-r)e^{\sigma(T_0-T_0)} < 1. We consider two cases, T0/τ = n + 1 and n < T0/τ < n + 1.

Case (I): Suppose T0/τ = n + 1.

From (3.5), we get that

\[
u(T_0) = u(0)((1-r)e^{\sigma(T_0-T_0)})e^{\sigma(T_0-n\tau)}.
\]  

(3.6)

From (3.6) and the third equation of system (3.4), we have

\[
u(T_0) = h((1-r)e^{\sigma(T_0-T_0)})e^{\sigma(T_0-n\tau)} \leq ET,
\]

and then

\[
\frac{h((1-r)e^{\sigma(T_0-T_0)})e^{\sigma(T_0-n\tau)}}{ET} \leq 1 \quad \text{or} \quad n \geq n^* = 1 + \frac{\ln(ETe^{-\sigma(T_0-n\tau)})}{\ln((1-r)e^{\sigma(T_0-T_0)})}.
\]

Case (II): Suppose n < T0/τ < n + 1.

Similarly, we can obtain

\[
\frac{h((1-r)e^{\sigma(T_0-T_0)})e^{\sigma(T_0-n\tau)}}{ET} \leq 1 \quad \text{or} \quad n \geq n^* = \frac{\ln(ETe^{-\sigma(T_0-n\tau)})}{\ln((1-r)e^{\sigma(T_0-T_0)})}.
\]

(3.7)

here \([T0/\tau] = n\) denotes integer partition of T0/τ.

**Theorem 3.1.** Suppose (1-r)e^{\sigma(T_0-T_0)} < 1, where σ = βL/(1+2λ) – (d+ω) > 0.

(i) When T0/τ = n + 1, if

\[
R_2 = \frac{h((1-r)e^{\sigma(T_0-n\tau)})e^{\sigma(T_0-n\tau)}}{ET} \leq 1, \quad \text{or} \quad n \geq n^* = 1 + \frac{\ln(ETe^{-\sigma(T_0-n\tau)})}{\ln((1-r)e^{\sigma(T_0-n\tau)})},
\]

then (3.4) has a solution which satisfies the boundary value problem.

(ii) When n < T0/τ < n + 1, if

\[
R_2 = \frac{h((1-r)e^{\sigma(T_0-n\tau)})e^{\sigma(T_0-n\tau)}}{ET} \leq 1, \quad \text{or} \quad n \geq n^* = \frac{\ln(ETe^{-\sigma(T_0-n\tau)})}{\ln((1-r)e^{\sigma(T_0-n\tau)})},
\]

then (3.4) has a solution which satisfies the boundary value problem.

Using the comparison theorem of impulsive differential equations, we can obtain that if u(T0) ≤ ET, then l(T0) ≤ ET. This implies we can control the infected plants under an Economic Threshold (ET) at t = T0. Therefore, we give an important result as follows.

**Theorem 3.2.** Suppose (1-r)e^{\sigma(T_0-T_0)} < 1, where σ = βL/(1+2λ) – (d+ω) > 0.

(i) When T0/τ = n + 1, if

\[
R_2 = \frac{h((1-r)e^{\sigma(T_0-n\tau)})e^{\sigma(T_0-n\tau)}}{ET} \leq 1, \quad \text{or} \quad n \geq n^* = \frac{\ln(ETe^{-\sigma(T_0-n\tau)})}{\ln((1-r)e^{\sigma(T_0-n\tau)})},
\]

then the infected plants are controlled under an Economic Threshold (ET) at t = T0.

(ii) When n < T0/τ < n + 1, if

\[
R_2 = \frac{h((1-r)e^{\sigma(T_0-n\tau)})e^{\sigma(T_0-n\tau)}}{ET} \leq 1, \quad \text{or} \quad n \geq n^* = \frac{\ln(ETe^{-\sigma(T_0-n\tau)})}{\ln((1-r)e^{\sigma(T_0-n\tau)})},
\]

then the infected plants are controlled under an Economic Threshold (ET) at t = T0.

We next consider the ET strategy in the whole time interval. By (3.5), if (1-r)e^{\sigma(T_0-T_0)} < 1, there σ = βL/(1+2λ) – (d+ω) > 0, then \(\lim_{n \to \infty} u(T_0) = 0\), i.e. \(\lim_{n \to \infty} u(t) = 0\). By the comparison theorem of impulsive differential equations, we can obtain that \(\lim_{n \to \infty} u(t) = 0 < ET\).

Using Lemma 3.0, it follows from the second and fourth equations of (1.2) that

\[
\frac{dl}{dt} \leq \frac{\beta L}{2} (d+\omega) l(t), \quad t \neq n\tau, \\
l(t^+) = (1-r)l(t), \quad t = n\tau.
\]  

By using impulse differential inequalities, we have

\[
l(t) \leq l(n_1 + \tau) \prod_{n_1 \leq n < t} (1-r) \exp \left( \int_{n_1}^{t} -(d+\omega) \, ds \right)
\]

\[
+ \int_{n_1}^{t} \prod_{n_1 \leq n < s} (1-r) \exp \left( \int_{n_1}^{s} -(d+\omega) \, ds \right) \frac{\beta L}{2} \, ds
\]

\[
= l(n_1 + \tau) \prod_{n_1 \leq n < t} (1-r)^{n-1} e^{-(d+\omega)(t-n_1-1)} + \frac{\beta L}{2} \int_{n_1}^{t} \prod_{n_1 \leq n < s} (1-r)^{n-1} e^{-(d+\omega)(s-n_1-1)} (\frac{d+\omega}{d+\omega}) \, ds
\]

\[
= l(n_1) e^{\sigma(T_0-n_1-1)} e^{-(d+\omega)(t-n_1-1)} + \frac{\beta L}{2} \int_{n_1}^{t} \prod_{n_1 \leq n < s} (1-r)^{n-1} e^{-(d+\omega)(s-n_1-1)} (\frac{d+\omega}{d+\omega}) \, ds
\]

\[
= l(n_1) e^{\sigma(T_0-n_1-1)} e^{-(d+\omega)(t-n_1-1)} + \frac{\beta L}{2} \int_{n_1}^{t} \prod_{n_1 \leq n < s} (1-r)^{n-1} e^{-(d+\omega)(s-n_1-1)} (\frac{d+\omega}{d+\omega}) \, ds
\]

\[
\leq l(n_1) e^{\sigma(T_0-n_1-1)} e^{-(d+\omega)(t-n_1-1)} + \frac{\beta L}{2} \int_{n_1}^{t} \prod_{n_1 \leq n < s} (1-r)^{n-1} e^{-(d+\omega)(s-n_1-1)} (\frac{d+\omega}{d+\omega}) \, ds
\]

\[
\leq l(n_1) e^{\sigma(T_0-n_1-1)} e^{-(d+\omega)(t-n_1-1)} + \frac{\beta L}{2} \int_{n_1}^{t} \prod_{n_1 \leq n < s} (1-r)^{n-1} e^{-(d+\omega)(s-n_1-1)} (\frac{d+\omega}{d+\omega}) \, ds
\]

which implies

\[
\limsup_{t \to \infty} u(t) \leq \frac{\beta L}{2(d+\omega)} \left[ \frac{r}{1-r} \right]
\]

If \(\beta L/(d+\omega) (1-r)/(e^{d+\omega} - 1 + r) \leq ET\), i.e. \(\beta L/(d+\omega) ET/[1-r/(e^{d+\omega} - 1 + r)] \leq 1\), then we can control the infected plants under an Economic Threshold (ET) in an infinite time interval.
Theorem 3.3. Assume \((1-r)e^{\alpha t} < 1\) or \(R_0 = \frac{\beta L}{2(a+d+o)e^r} \left[ 1 - \frac{r}{a+d+o} - 1 + r \right] \leq 1\), then the infected plants are controlled under an Economic Threshold (ET) in an infinite time interval.

According to the above analysis for Theorems 3.2 and 3.3, it is clearly to see that the conditions of Theorem 3.2 can deduce the results of Theorem 3.3 but the conditions of Theorem 3.3 cannot deduce the results of Theorem 3.2.

3.3. Permanence

Theorem 3.4. If \(R_1 > 1\), then system (1.2) is permanent.

Proof. Suppose \(X(t) = (S(t), L(t))\) any solution of system (1.2) with \(X(0) > 0\). From Lemma 3.0, we have \(S(t) \leq L(t) \leq L\) for all \(t \geq 0\). Therefore, \(S(t) \geq (BL + M)\), \(S(t) \geq (BL + M)\) for \(t \neq n\), and \(S(t) \geq (BL + M)\) for \(t = n\). Therefore, there exists a constant \(m_1 = \min(\beta L + M)\) for \(t > 0\), and \(S(t) \geq (BL + M)\) for \(t \geq 0\). Thus only we need to find \(m_1 > 0\) such that \(l(t) \geq m_1\) for \(t \geq 0\).

From \(R_1 > 1\), we can choose \(e_1 > 0\) and \(m_2 > 0\) to be small such that \(\eta_0 = 1 - (1-r)\exp(-m_2^2)\) for all \(t \geq 0\), and \(\eta_1 > 0\) such that \(\eta_1 > 0\) for all \(t \geq 0\). Therefore, we have \(S(t) \geq z(t)\), where \(z(t)\) is the solution of

\[
\frac{dz}{dt} = -(\beta m_2^2 + m_2)z(t) + \alpha z(t), \quad t \neq n,
\]

\[
S(t) = S(t) + \rho, \quad t = n.
\]

So we have \(S(t) \geq z(t)\) and \(z(t) \to z(t)\), as \(t \to \infty\), where \(z(t)\) is the solution of

\[
\frac{dz}{dt} = -(\beta m_2^2 + m_2)z(t), \quad t \neq n,
\]

\[
z(t^+) = z(t) + \rho, \quad t = n,
\]

\[
S(t^+) = S(t^+) + \rho, \quad t = n.
\]

Therefore, there exists a \(t_1 > 0\) such that

\[
S(t) \geq z(t) > z(t^+),
\]

and

\[
\frac{dI}{dt} = \left[ \frac{\beta z(t^+)}{1 + 2\alpha z(t^+)} - (d + \omega) \right] I(t), \quad t \neq n,
\]

\[
I(t^+) = (1 - r)I(t), \quad t = n
\]

for \(t \geq t_1\). Let \(N \in \mathbb{Z}^+\) and \(N \geq t_1\). Integrating (3.8) on \((n, (n+1))\), \(n \geq N\), we have

\[
l(n + 1) \geq l(n) \exp \left( \int_{(n + 1)}^{(n + 1) + r} \left[ \frac{\beta z(t)}{1 + 2\alpha z(t)} - (d + \omega) \right] dt \right)
\]

\[
= l(n) \eta^n.
\]

Then \(l(N + n) \geq l(N + n) \eta^n \to \infty\) as \(n \to \infty\), which is a contradiction to the boundedness of \(l(t)\). Hence there exists a \(t_0 > 0\) such that \(l(t_0) \geq m_2^*\). If \(l(t_0) \geq m_2^*\) for all \(t \geq t_0\), and let \(m_2 = m_2^*\) then our aim is obtained. Otherwise, let \(t^* = \inf_{t > t_0} \{ l(t) < m_2^* \} \), there are two possible cases for \(t^*\).

Case (I): \(t^* = n_1 t + n_1 / 2 \in \mathbb{Z}^+\). Then \(l(t) \geq m_2^*\) for \(t \in \{ t \in (t_1, t^*) \} \) and \(l(t^*) < m_2^*\). By the definition of \(m_1\), we can have \(\beta m_1 \leq d + \omega < 0\) and choose \(a, b, c, d, e, f \in \mathbb{Z}_+\) such that

\[
(1-r)^n \exp \left[ \left( \frac{\beta m_1}{1 + 2\alpha m_1} - (d + \omega) \right) n_2 \right] \eta^n
\]

\[
> (1-r)^n \exp \left[ \left( \frac{\beta m_1}{1 + 2\alpha m_1} - (d + \omega) \right) (n_2 + 1) \right] \eta^n
\]

\[
> 1.
\]
Since \( l(t) \leq m_2 \) for \( t \in (t^*, (n_4+1)t) \), (3.11) holds on \([t^*, (n_4+1) + n_2 + n_3]t \), so we have

\[
I((n_4 + 1 + n_2 + n_3) t) \geq (1-r)^{n_3} m_2^x \exp \left\{ \left( n_2 + 1 \right) \left[ \frac{\beta m_1}{1 + 2m_1} - (d + \omega) \right] t \right\}.
\]

Thus, \( l((n_4 + 1 + n_2 + n_3) t) \geq (1-r)^{n_3} m_2^x \exp ((n_2 + 1) \left[ \frac{\beta m_1}{1 + 2m_1} - (d + \omega) \right] t) > m_2^x \), which is a contradiction. Let \( T = \inf_{1 \leq r \leq \tau} (l(t) > m_2^x) \), then \( l(t) \leq m_2^x \) for \( t \in (t^*, T) \) and \( l(T) = m_2^x \). For \( t \in (T, \tau) \), suppose \( t \in (at + (k-1)r - \alpha t, at + k r) \), \( k \in \mathbb{N}, k \leq 1 + n_2 + n_3 \), we have

\[
l(t) \geq (1-r)^{k-1} m_2^x \exp \left\{ k \left[ \frac{\beta m_1}{1 + 2m_1} - (d + \omega) \right] \right\} \geq (1-r)^{n_3 + n_2 + n_3} m_2^x \exp \left\{ \left( n_2 + n_3 \right) \left[ \frac{\beta m_1}{1 + 2m_1} - (d + \omega) \right] t \right\}.
\]

Let \( m_2^x = (1-r)^{n_3 + n_2 + n_3} m_2^x \exp (1 + n_2 + n_3) \left[ \frac{\beta m_1}{1 + 2m_1} - (d + \omega) \right] t < m_2^x \), hence, we have \( l(t) \geq m_2^x \) for \( t \in (t^*, T) \). For \( t > T \), the same arguments can be continued since \( l(T) \geq m_2^x \).

Case (ii): There exists a \( t \in (t^*, (n_4+1)t) \) such that \( l(t) > m_2^x \). Let \( t^* = \inf_{1 \leq r \leq \tau} (l(t) > m_2^x) \), then \( l(t) \leq m_2^x \) for \( t \in (t^*, t^*) \) and \( l(t^*) = m_2^x \). For \( t \in (t^*, t^*) \), (3.11) holds true, integrating (3.11) on \((t^*, t^*)\), we have

\[
l(t) \geq l(t^*) \exp \left\{ \frac{\beta m_1}{1 + 2m_1} - (d + \omega) \right\} (t-t^*) \geq m_2^x \exp \left\{ \frac{\beta m_1}{1 + 2m_1} - (d + \omega) \right\} (t-t^*) > m_2^x \geq m_2^x.
\]

Since \( l(t^*) \geq m_2^x \) for \( t > t^* \), the same arguments can be continued. Hence \( l(t) \geq m_2^x \) for all \( t \geq t^* \). The proof is completed.

4. Positive periodic solution and bifurcation

By Theorems 3.0 and 3.1, we know that if \( R_1 < 1 \), then the disease-free periodic solution \( S(t,0) \) of system (1.2) is stable, and if \( R_1 > 1 \) then the disease-free periodic solution \( S(t,0) \) of system (1.2) is not stable and the system is permanent. In the following, we shall study the loss of stability phenomenon and prove that it is due to the onset of nontrivial periodic solutions obtained via a supercritical bifurcation in the limiting case, that is, \( R_1 > 1 \). Hence, we shall employ a fixed point argument. We denote by \( \Phi(U_0) \) the solution of the (unperturbed) system consisting of the first two equations of (1.2) with the initial state \( U_0 = (u_0^0, u_0^0) \), and \( \Phi = (\Phi_1, \Phi_2) \). We define the mapping \( \Theta_1, \Theta_2 : \mathbb{R}^2 \to \mathbb{R}^2 \) by

\[
\Theta_1(x, y) = x + \rho, \quad \Theta_2(x, y) = (1-r)x,
\]

and the mapping \( F_1, F_2 : \mathbb{R}^2 \to \mathbb{R}^2 \) by

\[
F_1(x, y) = \frac{\beta x_1 y_2}{1 + 2x_1} - \omega x_1 + \omega y_2, \quad F_2(x, y) = \frac{\beta x_1 y_2}{1 + 2x_1} - (d + \omega)y_2.
\]

Furthermore, let us define \( \Psi : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}^2 \) by

\[
\Psi(t, U_0) = \Theta(\Phi(t, U_0)), \quad \Psi(t, U_0) = (\Psi_1(t, U_0), \Psi_2(t, U_0)),
\]

where \( \Theta = (\Theta_1, \Theta_2) \). It is easy to see that \( \Psi \) is actually the stroboscopic mapping associated to the system (1.2), which puts in correspondence the initial data at \( 0^+ \) with the subsequent state of the system \( \Psi(t, U_0) \) at \( t^+ \), where \( t \) is the stroboscopic time snapshot. We reduce the problem of finding a periodic solution of (1.2) to a fixed problem. Here, \( U \) is a periodic solution of period \( \tau \) for (1.2) if and only if its initial value \( I(U) = U_0 \) is a fixed point for \( \Psi(t, \cdot) \). In order to establish the existence of nontrivial periodic solutions of (1.2), we need to prove the existence of the nontrivial fixed points of \( \Psi \).

Next, we deal with problem of the bifurcation of nontrivial periodic solution of system (1.2), near \( S(t,0) \). Assume that \( x_0 = (x_0, 0) \) is a starting point for the trivial periodic solution \( S(t,0) \), where \( x_0 = \frac{5}{3} \). To find a nontrivial periodic solution of period \( T \) with initial value \( X \), we need to solve the fixed point problem \( X = \Psi(T, X) \), or denoting \( T = \tau + T, X = X_0 + \tilde{X}, X_0 + \tilde{X} = \Psi(T + \tilde{T}, X_0 + \tilde{X}) \).

Define \(
N(\tilde{T}, \tilde{X}) = X_0 + \tilde{X} - \Psi(\tau + \tilde{T}, X_0 + \tilde{X}) = (N_1(\tilde{T}, \tilde{X}), N_2(\tilde{T}, \tilde{X})).
\)

At the fixed point \( N(\tilde{T}, \tilde{X}) = 0 \), we denote \( D_X N(0,0,0) = \begin{pmatrix} a_{00} & b_0 & 0 \\ c_0 & d_0 \end{pmatrix} \).

By formally deriving the equation

\[
\frac{d}{dt} \Phi(t, U_0) = F(\Phi(t, U_0)),
\]

This characterized the dynamics of the unperturbed flow associated to the first two equations in (1.2), one obtains that

\[
\frac{d}{dt} [D_X \Phi(t, U_0)] = D_X F(\Phi(t, U_0)) D_X \Phi(t, U_0).
\]
This relation will be integrated in what follows in order to compute the components of $D_x\Phi(t,X_0)$ explicitly. Firstly, it is clear that $\Phi(t,X_0) = (\Phi_1(t,X_0),0)$. Then we deduce that (4.2) takes the particular form

$$
\frac{d}{dt} \begin{bmatrix}
    \frac{\partial \Phi_1}{\partial x_1} \\
    \frac{\partial \Phi_2}{\partial x_2}
\end{bmatrix}(t,X_0) = \begin{bmatrix}
    -\mu - \frac{\beta S(t)}{1 + 2S(t)} d - \omega \\
    0
\end{bmatrix} \begin{bmatrix}
    \frac{\partial \Phi_1}{\partial x_1} \\
    \frac{\partial \Phi_2}{\partial x_2}
\end{bmatrix}(t,X_0).
$$

(4.3)

Then the initial condition for (4.3) at $t=0$ is

$$
D_x\Phi(0,X_0) = E_x,
$$

where $E_x$ is the identity matrix in $M_2(R)$. This implies the initial condition $rac{\partial \Phi_2(0,X_0)}{\partial x_1} = 0$. It follows that

$$
\frac{\partial \Phi_2(t,X_0)}{\partial x_1} = \exp\left(\int_0^t \left(\frac{\beta S(t)}{1 + 2S(t)} d - \omega\right) dt\right) \frac{\partial \Phi_2(0,X_0)}{\partial x_1} = 0, \quad \text{for } t \geq 0.
$$

(4.5)

From (4.3) one obtains that

$$
\frac{d}{dt} \left(\frac{\partial \Phi_1(t,X_0)}{\partial x_1}\right) = -\mu \left(\frac{\partial \Phi_1(t,X_0)}{\partial x_1}\right),
$$

$$
\frac{d}{dt} \left(\frac{\partial \Phi_2(t,X_0)}{\partial x_2}\right) = -\mu \frac{\partial \Phi_1(t,X_0)}{\partial x_2} + \left(\omega - \frac{\beta S(t)}{1 + 2S(t)} d - \omega\right) \frac{\partial \Phi_2(t,X_0)}{\partial x_2}.
$$

(4.1)

(4.2)

According to the initial condition, we have

$$
\frac{\partial \Phi_1(t,X_0)}{\partial x_1} = \exp(-\mu t),
$$

$$
\frac{\partial \Phi_2(t,X_0)}{\partial x_2} = \exp\left(\int_0^t \left(\omega - \frac{\beta S(s)}{1 + 2S(s)} d - \omega\right) ds\right).
$$

(4.4)

From (4.1), one obtains that

$$
D_x N(0,(0,0)) = E_x - BD_x\Phi(t,X_0).
$$

This implies

$$
D_x N(0,(0,0)) = \begin{bmatrix}
    a_0 & b_0 \\
    0 & d_0
\end{bmatrix},
$$

where

$$
a_0 = 1 - \exp(-\mu t) > 0,
$$

$$
b_0 = -\exp(-\mu t) \int_0^t \left(\omega - \frac{\beta S(s)}{1 + 2S(s)} d - \omega\right) ds,
$$

(4.3)

(4.4)

Fig. 4. The relationship between the parameters and $R_1$. (a) $R_1 - \tau$; (b) $R_1 - \tau$; (c) $R_1 - \tau$ and $r$; (d) $R_1 - \omega$. Suppose the unit of the plants density $S$ and $t$ is thousand and the unit of time is day.
obtain that the disease-free periodic solution \((\hat{S}(t),0)\) is locally asymptotically stable (in Fig. 2) if \(R_1 < 1\), and that the system (1.2) is permanent if \(R_1 > 1\). Therefore, \(R_1 = 1\) is a critical value. Using the bifurcation theorem, we show that once a threshold condition is reached, a stable nontrivial periodic solution emerges via a

\[ d_0 = 1-(1-(1-r)\exp\left(\int_0^\infty \left(\frac{\beta S(s)}{1+2S(s)} - d - \omega\right) ds\right). \]

The necessary condition for the bifurcation of nontrivial periodic solutions near \((\hat{S}(t),0)\) is then \(\mathcal{D}_0(N(0,0)) = 0\). Since \(\mathcal{D}_0(N(0,0))\) is an upper triangular matrix and 1-\(\exp(-\mu t) > 0\), it consequently follows that \(d_0 = 0\) i.e. \(R_1 = 1\) is necessary for the bifurcation. It now remains to show that this necessary condition is also sufficient. This assertion represents the statement of the following theorem, which is our main result.

**Theorem 4.1.** A supercritical bifurcation occurs at \(R_1 = 1\), in the sense that there is \(\epsilon > 0\) such that for all \(0 < \epsilon < \epsilon\) there is a stable positive nontrivial periodic solution of (1.2) with period \(t+\epsilon\).

Theorem 4.1 is further proved in Appendix A.

### 5. Numerical analysis and biological conclusions

Cultural control is one of the methods of IDM which is aimed at either curing or reducing the epidemic plant potential to reduce the damage or encouraging the healthy growth of the host plant. The goal of plant disease management is to reduce the economic damage caused by plant diseases. Plant disease management procedures are frequently determined by disease forecasting or disease modeling. It is very important to establish economic thresholds for the plant disease management scheme and to model approaches for predicting epidemics of plant disease (Jeger et al., 2004; Holt and Chancellor, 1997; Zhang and Holt, 2004).

Therefore, in the present paper we formulate two different plant disease models to estimate and analyze the effects of a cultural control strategy on disease control. We give the threshold \(R_0\) of the infected plants-eradication for system (1.1) with continuous cultural control and the threshold \(R_1\) of the infected plants-eradication for system (1.2) with impulsive cultural control. Moreover, we give also Economic Threshold \(R_2\) which implies that we do not need to eradicate the infected plants but to remove the infected plants under an Economic Threshold (ET). The results obtained here imply that we can control the period of control strategy application (\(R_1 < 1\)) such that the infected plant free periodic solution is logically stable. This means the infected plants could be completely eradicated if we implement the cultural control strategies relatively sparsely. The modeling methods and results obtained here partially extended and generalized Fishman’s works (Fishman and Marcus, 1984) on plant disease control, and all such models can be used to develop and investigate more realistic plant disease models. Furthermore, we obtain the Economic Thresholds \(R_2\) and \(R_3\) which mean not to eradicate the infected plants but to remove the infected plants under an Economic Threshold (ET) in a finite time and an infinite time.

At first, we formulate and analyze the plant disease model including continuous cultural control such as curing, replanting and/or removing diseased plants. The global asymptotical stability of disease-free equilibrium and positive equilibrium is investigated. The threshold \((R_0 < 1)\) of the infected plants-eradication for system (1.1) with continuous cultural control is obtained, which is also demonstrated in Figs. 1a and b, here \(R_0 = 0.8333 < 1\). This means the infected plants could be completely eradicated if we implement the cultural control strategies relatively sparsely. Fig. 1c shows that when \(R_0 = 1.250 > 1\), positive equilibrium \(E^*(S^*,I^*)\) is globally asymptotically stable. Secondly, we have developed and extended the epidemic model with continuous cultural control to include the impulsive cultural control strategy such as replanting and/or removing diseased plants. We investigate the stability of the infected plant free periodic solution and the permanence of system (1.2). From Theorems 3.0 and 3.3, we

Fig. 5. Illustration of basic behavior of solutions of the model (1.2), where parameters are fixed as follows \(\rho = 1, \beta = 1.2, \sigma = 1, \mu = 0.2, \omega = 0.1, d = 0.3, r = 0.5, \tau = 1.5\) the initial value is \((S_0, I_0) = (0.5, 0.3)\). (a) time series of \(S(t)\); (b) time series of \(I(t)\); (c) phase portrait of \(S(t)\) and \(I(t)\). Suppose the unit of the plants density \(S\) and \(I\) is thousand and the unit of time is day.
supercritical bifurcation, which is confirmed in Fig. 5. If keeping model (1.1)'s notations, \( \rho \) should be substituted by \( \rho_T \) in model (1.2). Similarly, \( r \) should be substituted by \( 1-e^{-\tau} \). This would then allow one to compare the two models. From Figs. 1 and 2, although \( \tau = 0.5 \) in Fig. 2 under the other same parameters, we can compute the \( R_0 = 0.83339 < 1 \) and \( R_1 = 0.83339 < 1 \), which theoretically shows impulsive removing diseased plants is more efficient and more economical than continuous removing. Theorem 3.2 means that if \( R_0 \leq 1 \) or pulse times \( n \geq n^* \), then the infected plants are controlled under an Economic Threshold (ET) at fixed time \( \tau = 0 \), which is also demonstrated in Fig. 3. In Fig. 1b, when \( R_1(x) = \alpha T \) then \( n^* = 5 \) and if \( r = 0.4 \), then \( n^* = 7 \). Thus proves that when the infected plants are controlled under an Economic Threshold (ET) at fixed time \( \tau = 0 \), the amount of one time impulsive removing diseased plants is larger, the impulsive removing times are fewer. In Fig. 4, we can easily see that \( R_1 \) is directly proportional to \( \tau \) value and inversely proportional to the \( r \) \( \& \) \( \alpha \) values, which implies to reduce the impulsive period of removing diseased plants or increase the amount of impulsive removing diseased plants or increase the amount of cured plants in order to control the infected plants under an Economic Threshold. Theorems 3.3 and 4.1 show the system (1.2) is permanent and has positive periodic solution, here \( R_1 = 1.070 > 1 \). This is confirmed in Fig. 5.

In conclusion, we give a mathematical conclusion as follows:

(I) Impulsive removing diseased plants is more efficient and more economical than continuous removing.

(II) One can appropriately control the amount of impulsive removing diseased plants and the impulsive period of removing to control infected plants under an Economic Threshold in a finite time and an infinite time.

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Appendix A

The proof of Theorem 4.1. According to the above notations, we obtain that

\[
\dim(\ker[D_X N(0,0,0)}) = 1
\]

and a basis \([-b_0/a_0, 1]\) in \(\ker[D_X N(0,0,0)]\). Then the equation \(N(\tilde{\tau},\tilde{X}) = 0\) is equivalent to \(N_1(\tilde{\tau},\tilde{\gamma}Y_0 + w(\tilde{\tau},\tilde{\gamma})E_0) = 0, N_2(\tilde{\tau},\tilde{\gamma}Y_0 + w(\tilde{\tau},\tilde{\gamma})E_0) = 0\), here \(E_0 = (1,0), Y_0 = (-b_0/a_0, 1), \tilde{X} = \gamma Y_0 + w E_0\) represents the direct sum decomposition of \(X\) using the projections onto \(\ker[D_X N(0,0,0)]\) (the central manifold) and \(\mathrm{Im}[D_X N(0,0,0)]\) (the stable manifold).

Define

\[
f_1(\tilde{\tau},\tilde{\gamma},w) = N_1(\tilde{\tau},\tilde{\gamma}Y_0 + w E_0), f_2(\tilde{\tau},\tilde{\gamma},w) = N_2(\tilde{\tau},\tilde{\gamma}Y_0 + w E_0).
\]

Then

\[
\frac{\partial f_1}{\partial \tau}(0,0,0) = \frac{\partial N_1}{\partial \frac{\tau}{\tilde{\tau}}}(0,0,0) = \tilde{a}_0 \neq 0.
\]

Hence, by the implicit function theorem, one may solve the equation \(f_1(\tilde{\tau},\tilde{\gamma},w) = 0\) near \((0,0,0)\) with respect to \(w\) as a function of \(\tilde{\tau} \& \tilde{\gamma}\), and find \(w = w(\tilde{\tau},\tilde{\gamma})\) such that \(w(0,0) = 0\) and

\[
f_1(\tilde{\tau},\tilde{\gamma};w(\tilde{\tau},\tilde{\gamma})) = N_1(\tilde{\tau},\tilde{\gamma}Y_0 + w(\tilde{\tau},\tilde{\gamma})E_0) = 0.
\]

Hence, we have

\[
\frac{\partial w}{\partial \tilde{\tau}}(0,0) = \left(\frac{\partial N_1(0,0)}{\partial \tilde{\tau}}\right)^{-1} \frac{\partial N_1(0,0)}{\partial \tilde{\gamma}} + \frac{b_0}{a_0} = 0.
\]

Then \(N(\tilde{\tau},\tilde{X}) = 0\) if and only if

\[
f_2(\tilde{\tau},\tilde{\gamma}) = N_2(\tilde{\tau},\left(-\frac{b_0}{a_0} \gamma + w(\tilde{\tau},\tilde{\gamma})\right)) = 0. \tag{A.1}
\]

Eq. (A.1) is called the determining equation and the number of its solutions are equal to the number of periodic solutions of (1.2). Denote

\[
g(\tilde{\tau},\tilde{\gamma}) = f_2(\tilde{\tau},\tilde{\gamma},w). \tag{A.2}
\]

It is clear to see that \(g(0,0) = N_2(0,0,0) = 0\). We determine the Taylor expansion of \(g\) around \(0\). For this, we compute the first order partial derivatives \(eg/\partial \tilde{\tau}(0,0)\) and \(eg/\partial \tilde{\gamma}(0,0)\).

The first order partial derivative of \(g\) with respect to \(\tilde{\tau}\) is

\[
\frac{\partial g}{\partial \tilde{\tau}}(\tilde{\tau},\tilde{\gamma}) = \frac{\partial}{\partial \tilde{\tau}}\left(\gamma - \frac{\partial}{\partial \tilde{\tau}}\frac{\partial N_1(0,0)}{\partial \tilde{\tau}} + \frac{\partial N_1(0,0)}{\partial \tilde{\gamma}} \left(-\frac{b_0}{a_0} \gamma + w(\tilde{\tau},\tilde{\gamma})\right)\right) E_0 = 0
\]

Note that

\[
\frac{\partial g}{\partial \tilde{\tau}}(\tilde{\tau},\tilde{\gamma}) = \frac{\partial}{\partial \tilde{\tau}}\left(\gamma - \frac{\partial}{\partial \tilde{\tau}}\frac{\partial N_1(0,0)}{\partial \tilde{\tau}} + \frac{\partial N_1(0,0)}{\partial \tilde{\gamma}} \left(-\frac{b_0}{a_0} \gamma + w(\tilde{\tau},\tilde{\gamma})\right)\right) E_0 = 0.
\]

Note that

\[
\frac{\partial g}{\partial \tilde{\tau}}(\tilde{\tau},\tilde{\gamma}) = \frac{\partial}{\partial \tilde{\tau}}\left(\gamma - \frac{\partial}{\partial \tilde{\tau}}\frac{\partial N_1(0,0)}{\partial \tilde{\tau}} + \frac{\partial N_1(0,0)}{\partial \tilde{\gamma}} \left(-\frac{b_0}{a_0} \gamma + w(\tilde{\tau},\tilde{\gamma})\right)\right) E_0 = 0.
\]

Then we have \(eg/\partial \tilde{\gamma}(0,0) = 0\). Similarly, the first order partial derivative of \(g\) with respect to \(\tilde{\tau}\) is

\[
\frac{\partial g}{\partial \tilde{\tau}}(\tilde{\tau},\tilde{\gamma}) = \frac{\partial}{\partial \tilde{\tau}}\left(\gamma - \frac{\partial}{\partial \tilde{\tau}}\frac{\partial N_1(0,0)}{\partial \tilde{\tau}} + \frac{\partial N_1(0,0)}{\partial \tilde{\gamma}} \left(-\frac{b_0}{a_0} \gamma + w(\tilde{\tau},\tilde{\gamma})\right)\right) E_0 = 0.
\]

Note that

\[
\frac{\partial g}{\partial \tilde{\tau}}(\tilde{\tau},\tilde{\gamma}) = \frac{\partial}{\partial \tilde{\tau}}\left(\gamma - \frac{\partial}{\partial \tilde{\tau}}\frac{\partial N_1(0,0)}{\partial \tilde{\tau}} + \frac{\partial N_1(0,0)}{\partial \tilde{\gamma}} \left(-\frac{b_0}{a_0} \gamma + w(\tilde{\tau},\tilde{\gamma})\right)\right) E_0 = 0.
\]

Hence, one obtains \(eg/\partial \tilde{\gamma}(0,0) = 0\).

Second partial derivatives of \(g\)

Denote

\[
A = \frac{\partial^2 g}{\partial \tilde{\tau}^2}(0,0), \quad B = \frac{\partial^2 g}{\partial \tilde{\tau} \partial \tilde{\gamma}}(0,0), \quad C = \frac{\partial^2 g}{\partial \tilde{\gamma}^2}(0,0).
\]

Take

\[
\xi_1(\tilde{\tau}) = \tilde{\tau} + \tilde{\tau}, \quad \xi_2(\tilde{\tau},\tilde{\gamma}) = \gamma - \frac{b_0}{a_0} \gamma + w(\tilde{\tau},\tilde{\gamma}), \quad \xi_3(\gamma) = \gamma.
\]

Now we calculate these quantities in terms of the parameters of the equation.
Firstly, we have

\[
\frac{\partial^2 g}{\partial T^2}(\tilde{T}, \tilde{y}) = \frac{\partial^2}{\partial T^2} \left[ \tilde{\zeta}_2 - \theta_2 \Phi(\zeta_1, \zeta_2, \zeta_3)(\tilde{T}, \tilde{y}) \right]
\]

\[
= -\frac{\partial^2 \theta_1}{\partial \tilde{y}^2} \left( \frac{\partial \Phi_1(\zeta_1, \zeta_2, \zeta_3)}{\partial \tilde{y}} + \frac{\partial \Phi_1(\zeta_1, \zeta_2, \zeta_3)}{\partial \tilde{x}_1} \frac{\partial \tilde{y}}{\partial \tilde{x}_1} \right)^2
\]

\[
- \frac{\partial^2 \theta_2}{\partial \tilde{y}^2} \left( \frac{\partial \Phi_1(\zeta_1, \zeta_2, \zeta_3)}{\partial \tilde{y}} + \frac{\partial \Phi_1(\zeta_1, \zeta_2, \zeta_3)}{\partial \tilde{x}_1} \frac{\partial \tilde{y}}{\partial \tilde{x}_1} \right)^2
\]

\[
- \frac{\partial^2 \theta_2}{\partial \tilde{x}_1 \partial \tilde{x}_2} \left( \frac{\partial \Phi_1(\zeta_1, \zeta_2, \zeta_3)}{\partial \tilde{x}_1} \frac{\partial \tilde{y}}{\partial \tilde{x}_1} + \frac{\partial \Phi_1(\zeta_1, \zeta_2, \zeta_3)}{\partial \tilde{x}_2} \frac{\partial \tilde{y}}{\partial \tilde{x}_2} \right)^2
\]

\[
- \frac{\partial \theta_2}{\partial \tilde{x}_1} \left( \frac{\partial \Phi_1(\zeta_1, \zeta_2, \zeta_3)}{\partial \tilde{x}_1} \frac{\partial \tilde{y}}{\partial \tilde{x}_1} + \frac{\partial \Phi_1(\zeta_1, \zeta_2, \zeta_3)}{\partial \tilde{x}_2} \frac{\partial \tilde{y}}{\partial \tilde{x}_2} \right)^2
\]

\[
- \frac{\partial \theta_2}{\partial \tilde{x}_1} \left( \frac{\partial \Phi_1(\zeta_1, \zeta_2, \zeta_3)}{\partial \tilde{x}_1} \frac{\partial \tilde{y}}{\partial \tilde{x}_1} + \frac{\partial \Phi_1(\zeta_1, \zeta_2, \zeta_3)}{\partial \tilde{x}_2} \frac{\partial \tilde{y}}{\partial \tilde{x}_2} \right)^2
\]

\[
+ 2 \frac{\partial \Phi_1(\zeta_1, \zeta_2, \zeta_3)}{\partial \tilde{x}_1} \frac{\partial \tilde{y}}{\partial \tilde{x}_1} \frac{\partial \Phi_1(\zeta_1, \zeta_2, \zeta_3)}{\partial \tilde{x}_2} \frac{\partial \tilde{y}}{\partial \tilde{x}_2}
\]

\[
- 2 \frac{\partial \theta_2}{\partial \tilde{x}_1} \left( \frac{\partial \Phi_1(\zeta_1, \zeta_2, \zeta_3)}{\partial \tilde{x}_1} \frac{\partial \tilde{y}}{\partial \tilde{x}_1} + \frac{\partial \Phi_1(\zeta_1, \zeta_2, \zeta_3)}{\partial \tilde{x}_2} \frac{\partial \tilde{y}}{\partial \tilde{x}_2} \right)^2
\]

\[
+ 2 \frac{\partial \Phi_1(\zeta_1, \zeta_2, \zeta_3)}{\partial \tilde{x}_1} \frac{\partial \tilde{y}}{\partial \tilde{x}_1} \frac{\partial \Phi_1(\zeta_1, \zeta_2, \zeta_3)}{\partial \tilde{x}_2} \frac{\partial \tilde{y}}{\partial \tilde{x}_2}
\]

Since \(\partial w(0,0) / \partial y = 0\), then \(C = (1 - r) \left( 2b_0 / a_0 \right) \left( \partial^2 \Phi_2(\tau, X_0) / \partial x_1 \partial x_2 \right)\). Therefore, for determining \(C\), one needs to calculate the terms \(\partial^2 \Phi_2(\tau, X_0) / \partial x_1 \partial x_2\) and \(\partial^2 \Phi_2(\tau, X_0) / \partial x_2^2\).

We calculate:

\[
\frac{d}{dt} \left( \frac{\partial^2 \Phi_2(t, X_0)}{\partial x_1 \partial x_2} \right) = \left( \frac{\beta \bar{S}(t)}{1 + \gamma S(t)} \right) \left( -d - \omega \right) \frac{\partial^2 \Phi_2(t, X_0)}{\partial x_1 \partial x_2}
\]

\[
+ \frac{\beta \bar{S}(t)}{1 + \gamma S(t)} \frac{\partial \Phi_1(t, X_0)}{\partial x_1} \frac{\partial \Phi_2(t, X_0)}{\partial x_2}.
\]

Note that

\[
\frac{\partial^2 \Phi_2(0, X_0)}{\partial x_1 \partial x_2} = 0.
\]

Hence we have

\[
\frac{\partial^2 \Phi_2(t, X_0)}{\partial x_2^2} = \exp \left( \int_0^t \left( \frac{\beta \bar{S}(s)}{1 + \gamma S(s)} \right) (d + \omega) ds \right)
\]

\[
\times \int_0^t \left( \frac{\beta \bar{S}(s)}{1 + \gamma S(s)} \right)^2 \frac{\partial \Phi_1(s, X_0)}{\partial x_1} \frac{\partial \Phi_2(s, X_0)}{\partial x_2} ds.
\]

Similarly, we have

\[
\frac{d}{dt} \left( \frac{\partial^2 \Phi_2(t, X_0)}{\partial x_2^2} \right) = \left( \frac{\beta \bar{S}(t)}{1 + \gamma S(t)} \right) \left( -d - \omega \right) \frac{\partial^2 \Phi_2(t, X_0)}{\partial x_2^2}
\]

\[
+ \frac{\beta \bar{S}(t)}{1 + \gamma S(t)} \frac{\partial \Phi_1(t, X_0)}{\partial x_1} \frac{\partial \Phi_2(t, X_0)}{\partial x_2}.
\]

And note that

\[
\frac{\partial^2 \Phi_2(0, X_0)}{\partial x_2^2} = 0.
\]

Hence we have

\[
\frac{\partial^2 \Phi_2(t, X_0)}{\partial x_2^2} = \exp \left( \int_0^t \left( \frac{\beta \bar{S}(s)}{1 + \gamma S(s)} \right) (d + \omega) ds \right)
\]

\[
\times \int_0^t \left( \frac{\beta \bar{S}(s)}{1 + \gamma S(s)} \right)^2 \frac{\partial \Phi_1(s, X_0)}{\partial x_1} \frac{\partial \Phi_2(s, X_0)}{\partial x_2} ds.
\]

Substituting (A.3) and (A.4) into C yields

\[
C = (1 - r) \left[ 2b_0 a_0 \right] \frac{\partial \Phi_2(t, X_0)}{\partial x_1 \partial x_2}
\]

\[
= (1 - r) \left[ 2b_0 a_0 \exp \left( \int_0^t \left( \frac{\beta \bar{S}(s)}{1 + \gamma S(s)} \right) (d + \omega) ds \right) \right]
\]

\[
\times \int_0^t \left( \frac{\beta \bar{S}(s)}{1 + \gamma S(s)} \right)^2 \frac{\partial \Phi_1(s, X_0)}{\partial x_1} ds
\]

\[
- \exp \left( \int_0^t \left( \frac{\beta \bar{S}(s)}{1 + \gamma S(s)} \right) (d + \omega) ds \right) \cdot \int_0^t \left( \frac{\beta \bar{S}(s)}{1 + \gamma S(s)} \right)^2 ds.
\]
Similarly, we also have
\[
B = - (1 - r) \left[ \frac{\partial^2 \Phi_2 (x, \tau)}{\partial x_2^2} \right]_{x_0} \frac{\partial \Phi_1 (x, 0)}{\partial t} + \frac{\partial^2 \Phi_2 (x, \tau)}{\partial t^2} \left[ \frac{\partial \Phi_1 (x, 0)}{\partial x} \right]_{x_0} \\
= - (1 - r) \left[ \exp \left( \int_0^T \left( \frac{\beta S(s)}{1 + zS(s)} (d + \omega) \right) ds \right) \\
\times \left( 1 + zS(r) \right) \right] \frac{\partial \Phi_1 (x, 0)}{\partial t} \frac{\partial S(r)}{\partial x} \left( 1 + zS(r) \right) \\
+ \left( \frac{\beta S(r)}{1 + zS(r)} (d + \omega) \right) \exp \left( \int_0^T \left( \frac{\beta S(s)}{1 + zS(s)} (d + \omega) \right) ds \right) \\
= (1 - r) \left[ - \exp \left( \int_0^T \left( \frac{\beta S(s)}{1 + zS(s)} (d + \omega) \right) ds \right) \\
\times \left( 1 + zS(r) \right) \right] \frac{\partial \Phi_1 (x, 0)}{\partial t} \frac{\partial S(r)}{\partial x} \left( 1 + zS(r) \right) \\
- \left( \frac{\beta S(r)}{1 + zS(r)} (d + \omega) \right) \exp \left( \int_0^T \left( \frac{\beta S(s)}{1 + zS(s)} (d + \omega) \right) ds \right) .
\]

Since \( R_1 = 1 \), i.e.
\[
\int_0^T \frac{\beta S(r)}{1 + zS(r)} ds = (d + \omega) T.
\]
And since
\[
\frac{\partial \Phi_1}{\partial x} < 0, \quad \frac{\beta S(r)}{1 + zS(r)} < \frac{1}{T} \int_0^T \frac{\beta S(s)}{1 + zS(s)} ds,
\]
then one obtains \( B > 0 \).

Then we have
\[
A = \frac{\partial^2 g}{\partial x^2} (0, 0) = 0, \quad B = \frac{\partial^2 g}{\partial t^2} (0, 0) > 0, \quad C = \frac{\partial^2 g}{\partial x \partial t} (0, 0),
\]
and then
\[
g(\bar{T}, \gamma) = B/\bar{T} + C/\bar{T}^2 + O(\bar{T}, \gamma)(\bar{T}^2 + \gamma^2).
\]
By denoting \( \bar{T} = h \gamma \) (where \( h = h(\gamma) \)), we obtain that \( (A.1) \) is equivalent to
\[
Bh + C \frac{h^2}{2} + O(h, \gamma)(1 + h^2) = 0. \quad (A.5)
\]
In the following, we discuss the solutions of Eq. \( (A.5) \) with respect to \( h \) to consider two cases:

Case I: If \( C = \frac{\partial^2 g}{\partial x \partial t} (0, 0) > 0 \), by denoting \( \bar{T} = h \gamma \) (where \( h = h(\gamma) \)), we obtain that \( (A.1) \) is equivalent to
\[
C \frac{h^2}{2} + Bh + O(h, \gamma)(1 + h^2) = 0. \quad (A.6)
\]
Since \( B > 0 \), Eq. \( (A.6) \) is solvable with respect to \( h \) as a function of \( \gamma \). Moreover, here \( h \approx -2B/C < 0 \).

Case II: If \( C = \frac{\partial^2 g}{\partial x \partial t} (0, 0) < 0 \), by denoting \( \bar{T} = h \gamma \) (where \( h = h(\gamma) \)). Similarly we have \( h \approx -2B/C > 0 \).

This means that there is a supercritical bifurcation to a nontrivial periodic solution near a period \( \tau \) which satisfies the sufficient condition of the bifurcation for \( R_1 = 1 \).

It is noteworthy that since this periodic solution appears via a supercritical bifurcation, the nontrivial periodic solution is stable. Therefore, there exists \( \varepsilon > 0 \) such that for all \( 0 < \gamma < \varepsilon \) there is a stable positive nontrivial periodic solution of (1.2) with period \( \pi + \bar{T}(\gamma) \) which starts in \( x_0, y_0 + w(\bar{T}(\gamma), \gamma) \). Here, \( x_0, y_0, \pi, w, \bar{T} \) are as defined above. The proof is completed. □

References


